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Max-Min Control Problems: A System Theoretic Approach

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***Abstract*—In this paper a "max-min controllability" concept for a situation in which two linear control systems are in conflict is introduced and characterized. This concept is employed in solving a max-min linear-quadratic control problem with terminal state constraints and the relationship with differential game theory is discussed.**

I. Introduction

CONSIDER the following linear system with dual controls:

$$\dot{x}=A(t)x+B_p(t)u+B_e(t)v, \qquad x(t_0)=x_0. \quad (1.1)$$

Here $x=x(t)$ is the state vector in Euclidean space R^n, with x_0 a specified initial state at time t_0. The vectors $u=u(t)\in R^{m_p}$ and $v=v(t)\in R^{m_e}$, regarded, respectively, as the pursuer and evader controls, are required to satisfy $\int_I\|u(t)\|^2dt<\infty$ and $\int_I\|v(t)\|^2dt<\infty$ on each compact interval $I\subset[t_0,\infty)$, where $\|\cdot\|$ denotes the Euclidean norm. The matrices A, B_p, and B_e are assumed to have entries which are real and measurable on $[t_0,\infty)$. For any pair of controls u and v we shall denote by $x(t)=\phi(t,t_0,x_0,u,v)$ the corresponding unique solution of (1.1) emanating from x_0 at time t_0 ($t\geqslant t_0$).

In situations in which the pursuer and evader are in competition, it is natural to seek a comparison between their control capabilities. Towards this end we introduce the following concepts.

Definition 1.1: An event (t_0,x_0) in system (1.1) is *strongly max-min controllable at time* T ($T\geqslant t_0$) if for each announced evader control v on $[t_0,T]$ there exists a pursuer control u on $[t_0,T]$ such that $x(T)=\phi(T,t_0,x_0,u,v)=0$. The event (t_0,x_0) is *strongly max-min controllable* if it is strongly max-min controllable for some $T\in[t_0,\infty)$.

Definition 1.2: An event (t_0,x_0) in system (1.1) is *weakly max-min controllable* if for each announced evader control v on $[t_0,\infty)$, there exists a time $\tilde{t}=\tilde{t}(v)\geqslant t_0$ and a pursuer control u on $[t_0,\tilde{t}]$ such that $x(\tilde{t})=\phi(\tilde{t},t_0,x_0,u,v)=0$.

Clearly strong max-min controllability of an event implies weak max-min controllability. That the converse is also true is not immediately evident since it is not clear that when weak max-min controllability holds there exists any one time T at which capture (i.e., $x(T)=0$) can be imposed by the pursuer in face of any evader control. This, however, is indeed the case as is shown in [1], and the two concepts of max-min controllability are actually equivalent. Henceforth, we will simply speak about *max-min controllability* referring to the simpler Definition 1.1.

It should be observed that max-min controllability generalizes the concept of controllability in linear control systems as expounded by Kalman (see, e.g., [2], [3]). While the existing "one player" controllability theory will be brought to bear on our development of the two player case, certain significant difficulties and interesting differences arise, as will be pointed out below.

Our results on max-min controllability will be employed in solving the following *restricted end-point max-min control problem*, denoted (P).

(P): We are given a linear dual control system (1.1) with $x_0\neq 0$. The evader announces a control function v, and the pursuer (if he has the capability) responds with a control function u such that $x(T)=0$, where $T>t_0$ is a prespecified time. The players' control choices are to be made in accordance with the optimization of the payoff functional

$$P(u,v)\triangleq\int_{t_0}^{T}\left[\|u(t)\|^2-\|v(t)\|^2\right]dt \qquad (1.2)$$

where it is understood that the evader is the maximizing player while the pursuer is the minimizer.

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Let $\Phi(t,t_0)$ denote the fundamental matrix solution corresponding to system (1.1) and define the vector function

$$z(t)=\Phi(t_0,t)x(t), \qquad t \geqslant t_0. \tag{1.3}$$

It is readily verified that z satisfies the following differential equation:

$$\dot{z}=\tilde{B}_p(t)u-\tilde{B}_e(t)v \quad z(t_0)=x(t_0)=x_0, \tag{1.4}$$

where $\tilde{B}_p(t) \triangleq \Phi(t_0,t)B_p(t)$ and $\tilde{B}_e(t)=-\Phi(t_0,t)B_e(t)$; and $z(t)=0$ if and only if $x(t)=0$. Hence, system (1.4) is completely equivalent in respect to max-min controllability to system (1.1).

A more general type of max-min control situation can be described by a pair of linear control systems:

$$\dot{x}_p=A_p(t)x_p+B_p(t)u, \qquad x_p(t_0)=x_{p0} \tag{1.5}$$

$$\dot{x}_e=A_e(t)x_e+B_e(t)v, \qquad x_e(t_0)=x_{e0} \tag{1.6}$$

where (1.5) represents the pursuer dynamics and (1.6) represents the evader dynamics. Both x_p and x_e are in R^n and capture is interpreted as an event $(t,x_p(t),x_e(t))$ such that $x_p(t)=x_e(t)$. Hence, max-min controllability for systems (1.5) and (1.6) refers to the existence of "capture events" as defined above, with Definitions 1.1 and 1.2 remaining otherwise intact. For a fixed $T>t_0$ define

$$z(t)=\Phi_p(T,t)x_p(t)-\Phi_e(T,t)x_e(t), \qquad t_0 \leqslant t \leqslant T. \tag{1.7}$$

With (1.7), (1.5) and (1.6) yield the following differential equation for z:

$$\dot{z}=\tilde{B}_p(t)u-\tilde{B}_e(t)v, \quad z(t_0)=\Phi_p(T,t_0)x_{p0}-\Phi_e(T,t_0)x_{e0} \tag{1.8}$$

where $\tilde{B}_p(t) \triangleq \Phi_p(T,t)B_p(t)$ and $\tilde{B}_e(t) \triangleq \Phi_e(T,t)B_e(t)$ for $t_0 \leqslant t \leqslant T$; and $\Phi_p(T,t)$ and $\Phi_e(T,t)$ are the fundamental matrix solutions for (1.5) and (1.6), respectively. It is readily seen that $x_p(T)=x_e(T)$ if and only if $z(T)=0$ and hence max-min controllability (in time T) for systems (1.5) and (1.6) is equivalent to max-min controllability (in time T) for (1.8).

Since system (1.8) is essentially the same as (1.4) we shall henceforth restrict our attention primarily to systems of the form (1.4) to which we shall refer as *standard*.

In [4] Ho, Bryson, and Baron applied the variational calculus to a linear-quadratic differential game of fixed duration (without terminal constraints). They then applied their results to problem (P) by means of penalty functions, and for a special case they obtained a solution as well as a sufficient condition for what we termed max-min controllability. While in the present paper we proceed from a system-theoretic rather than a variational viewpoint, our results on problem (P) are related to those in [4]. (In this regard see Remark 3.11 and also Section IV below.)

Finally, it was pointed out by a referee that problem (P) has various similarities with the so called "Stackelberg solution" of a game. In this regard, the interested reader is referred to [5] and [6].

The organization of the paper is as follows: In Section II we shall give an algebraic characterization of max-min controllability with some special attention to the autonomous case. In Section III problem (P) is solved, and in Section IV we compare our results with results and concepts concerning differential games with more elaborate information schemes as in Isaacs [7] and Hájek [8], [9], and we discuss the relation between max-min controllability and feedback.

II. Max-Min Controllability

For a standard system (1.4) defines the controllability Grammians for the pursuer and evader by

$$W_p(t_0,t) \triangleq \int_{t_0}^{t} \tilde{B}_p(\sigma)\tilde{B}_p'(\sigma)\,d\sigma, \qquad t \geqslant t_0 \tag{2.1}$$

$$W_e(t_0,t) \triangleq \int_{t_0}^{t} \tilde{B}_e(\sigma)\tilde{B}_e'(\sigma)\,d\sigma, \qquad t \geqslant t_0 \tag{2.2}$$

where the prime denotes transpose. Clearly W_p and W_e are symmetric nonnegative definite $(n \times n)$ matrices. Also, since for every pair F, G of symmetric nonnegative matrices $\mathfrak{R}(F) \subset \mathfrak{R}(F+G)$ (where $\mathfrak{R}(\cdot)$ denotes range), it can be readily verified that the rank of these Grammians is a nondecreasing and left-continuous function of time. For the one player case, i.e., $\tilde{B}_e \equiv 0$, it is well known that the pursuer can drive the event (t_0,z_0) to (t_1,z_1) in system (1.4) if and only if

$$z_0-z_1 \in \mathfrak{R}(W_p(t_0,t_1)). \tag{2.3}$$

This result generalizes in the two player case to the following algebraic condition for max-min controllability.

Theorem 2.1: Given system (1.4) with $z_0 \neq 0$, a necessary and sufficient condition for an event (t_0,z_0) to be max-min controllable in finite time $T>t_0$ is that the following conditions hold:

$$z_0 \in \mathfrak{R}(W_p(t_0,T)) \tag{2.4}$$

$$\mathfrak{R}(W_e(t_0,T)) \subset \mathfrak{R}(W_p(t_0,T)). \tag{2.5}$$

Proof: Max-min controllability in time T is equivalent to

$$z_0-\int_{t_0}^{T} \tilde{B}_e(t)v(t)\,dt \in \mathfrak{R}(W_p(t_0,T)) \tag{2.6}$$

for all evader controls v. Since

$$\bigcup_{v} \int_{t_0}^{T} \tilde{B}_e(t)v(t)\,dt = \mathfrak{R}(W_e(t_0,T)),$$

it follows that (2.6) is equivalent to

$$z_0+\mathfrak{R}(W_e(t_0,T)) \subset \mathfrak{R}(W_p(t_0,T)) \tag{2.7}$$

which in turn is equivalent to (2.4) and (2.5). □

In a one player linear control system (i.e., $\tilde{B}_e(t)=0$) it is well known that if an event (t_0,z_0) [in system (1.4)] can be steered to $(T,0)$, then it can also be steered to $(T',0)$ for all $T'>T$. In the two player case this is no longer true in the sense that the fact that an event (t_0,z_0) is max-min controllable in time T does *not* imply that the same is also true for all $T'>T$. Indeed, there may exist an interval $(t_1,t_2]$ (with $t_0<t_1<t_2<\infty$) such that (t_0,z_0) is never max-min controllable in time T for $T\notin(t_1,t_2]$. This fact is not surprising when one considers condition (2.5). It is then readily observed that max-min controllability has many similarities to (ordinary) controllability to a time-varying manifold, a situation which received essentially no attention in the literature. The above mentioned phenomenon is illustrated by the following simple example.

Example 2.2: Consider system (1.4) with dimensions $n=m_p=m_e=2$. Let

$$\tilde{B}_p(t)=\begin{pmatrix}0\\ b_p(t)\end{pmatrix} \quad \text{and} \quad \tilde{B}_e(t)=\begin{pmatrix}b_e(t)\\ 1\end{pmatrix}$$

where

$$b_p(t)=\begin{cases}0, & \text{for } t\in(0,1]\\ 1, & \text{for } t\in(1,\infty)\end{cases}$$

$$b_e(t)=\begin{cases}0, & \text{for } t\in(0,2]\\ 1, & \text{for } t\in(2,\infty).\end{cases}$$

It is easily seen that the initial event $\left(0,\begin{pmatrix}0\\1\end{pmatrix}\right)$ is max-min controllable at every $T\in I=(1,2]$ but never for $T\notin I$.

For the one player case it is immediate from (2.3) that for any t_0, the set of states z_0 such that the event (t_0,z_0) is controllable constitutes a subspace of R^n. The analog of this important geometric fact is also valid in the two player case, as we now show.

Theorem 2.3: Consider system (1.4). For any t_0, the set $C(t_0)$ of states z_0 such that the corresponding event (t_0,z_0) is max-min controllable is a linear subspace of R^n.

Remark 2.4: It is of course an immediate consequence of Theorem 2.1 that the set of states z_0 for which the corresponding event (t_0,z_0) is max-min controllable in (a fixed) time T, is a subspace. Theorem 2.3 claims that the set of states z_0 for which the event (t_0,z_0) is max-min controllable in *any* time T is also a subspace. This is less obvious especially when considering our preceding discussion in relation to the concept of *weak* max-min controllability.

Proof of Theorem 2.3: First observe that $0\in C(t_0)$ trivially. Also, if z_0 satisfies (2.4) for some T such that (2.5) holds, the same is also true for αz_0, $\alpha\in R$. Hence, $z_0\in C(t_0)$ implies that $\alpha z_0\in C(t_0)$ for every real α. Now let z_0^1 and z_0^2 be in $C(t_0)$ and let T^i denote the times satisfying (2.4) and (2.5) for z_0^i, $i=1,2$. If we assume (without loss of generality) that $T^2\geqslant T^1$ then $z_0^1+z_0^2\in \mathfrak{R}(W_p(t_0,T^2))$ and since (2.5) holds for T^2 it follows that $z_0^1+z_0^2\in C(t_0)$ and the proof is complete. □

Consider now the pair of systems (1.5) and (1.6) and assume that A_p, B_p, A_e, and B_e are constant matrices. Define the following "controllability matrices":

$$[A_p|B_p]\triangleq[B_p,A_pB_p,\cdots,A_p^{n-1}B_p] \tag{2.8}$$

$$[A_e|B_e]\triangleq[B_e,A_eB_e,\cdots,A_e^{n-1}B_e]. \tag{2.9}$$

We can then state the following result for autonomous systems (which has no analog in the time dependent case).

Theorem 2.5: Consider systems (1.5) and (1.6) with A_p, B_p, A_e, and B_e constant matrices. Assume that $x_{e0}\in \mathfrak{R}([A_e|B_e])$. Then a necessary and sufficient condition that for any t_0 the event (t_0,x_{p0},x_{e0}) be max-min controllable is that

$$x_{p0}\in\mathfrak{R}([A_p|B_p]) \tag{2.10}$$

and

$$\mathfrak{R}([A_e|B_e])\subset\mathfrak{R}([A_p|B_p]). \tag{2.11}$$

Moreover, if (t_0,x_{p0},x_{e0}) is max-min controllable, then it is max-min controllable in time T for every $T>t_0$.

Proof: Max-min controllability is equivalent to the existence of a time $T>t_0$ such that at T the reachable set of the pursuer system contains the reachable set of the evader system. Using the standard exponential notation, this is equivalent to

$$e^{A_eT}x_{e0}+\mathfrak{R}([A_e|B_e])\subset e^{A_pT}x_{p0}+\mathfrak{R}([A_p|B_p]) \tag{2.12}$$

which in turn is equivalent to (2.11) along with

$$e^{A_pT}x_{p0}-e^{A_eT}x_{e0}\in\mathfrak{R}([A_p|B_p]). \tag{2.13}$$

Now, $x_{e0}\in\mathfrak{R}([A_e|B_e])$ if and only if $e^{A_eT}x_{e0}\in\mathfrak{R}([A_e|B_e])$, and again applying the A-invariance property, but to the pursuer system, we conclude that (2.13) is equivalent to (2.10). Since $T>t_0$ can be chosen arbitrarily the proof is complete. □

Theorem 2.5 states essentially that if the evader initial state is evader-controllable (in the evader control system) then max-min controllability holds if and only if the pursuer initial state is pursuer-controllable and the evader reachable subspace (from the origin) is also pursuer-reachable. In case $A_p=A_e$ we can subtract (1.6) from (1.5) letting $x=x_p-x_e$. This gives us system (1.1) with B_p remaining intact and B_e replaced by $-B_e$. In this case we can state the following corollary to Theorem 2.5, which is a specialization of Theorem 2.1 to the autonomous case.

Corollary 2.6: Consider system (1.1) with A, B_p and B_e constant matrices. A necessary and sufficient condition for max-min controllability of an event (t_0,x_0) (for every t_0) is

$$x_0\in\mathfrak{R}([A|B_p]) \tag{2.12}$$

and

$$\mathfrak{R}([A|B_e])\subset\mathfrak{R}([A|B_p]) \tag{2.13}$$

in which case x_0 can be steered to the origin in arbitrarily short time in the presence of any (announced) evader control.

Remark 2.7: While condition (2.13) is clearly satisfied whenever $\mathcal{R}(B_e)\subset\mathcal{R}(B_p)$, this condition is by no means necessary. However, in cases where (2.13) holds and $\mathcal{R}(B_e)\not\subset\mathcal{R}(B_p)$ we will see later that our information scheme is extremely restrictive and the evader may gain significant advantages by employing a feedback control rather than predesignated controls. This interesting situation is further discussed in Section IV.

III. Solution of Problem (P)

In this section we will focus our attention on the fixed duration restricted end-point max-min control problem (P) which was introduced in Section I. We now rephrase problem (P) as follows.

(P)*:* Given system (1.4) with $z_0\neq 0$, let $T>t_0$ be such that (2.4) and (2.5) are satisfied, i.e., z_0 is max-min controllable in time T. The evader announces a control v and the pursuer responds with a control u such that the associated solution of (1.4) satisfies $z(T)=0$, while both players make their control choices in accordance with the optimization (evader maximizing, pursuer minimizing) of the payoff functional $P(u,v)$ given by (1.2).

Let the evader specify a control v on $[t_0,T]$. Then

$$\int_{t_0}^{T}\tilde{B}_e(t)v(t)\,dt\in\mathcal{R}[W_e(t_0,T)] \tag{3.1}$$

and there exists $y\in R^n$ such that

$$\int_{t_0}^{T}\tilde{B}_e(t)v(t)\,dt=W_e(t_0,T)y. \tag{3.2}$$

Due to (2.4) and (2.5)

$$\begin{aligned} z(T)=0=z_0+\int_{t_0}^{T}\tilde{B}_p(t)u(t)\,dt-W_e(t_0,T)y\\ =z_0-W_p(t_0,T)w-W_e(t_0,T)y \end{aligned} \tag{3.3}$$

for some $w=w(y)\in R^n$. In view of the well-known minimum energy law (see, e.g., [2]) any pursuer control

$$u_y(t)=-\tilde{B}_p'w \tag{3.4}$$

will drive z_0 to 0 and minimize $P(u,v)$ against the given v. Furthermore, as is easily verifiable,

$$P(u_y,v)=w'W_p(t_0,T)w-\int_{t_0}^{T}\|v(t)\|^2dt. \tag{3.5}$$

Assuming temporarily that y is specified, a reapplication of the minimum energy rule for the evader gives

$$v_y(t)=\tilde{B}_e'(t)y \tag{3.6}$$

as evader optimal control for a given y. Furthermore, for the fixed choice of y, $P(u_y,v_y)$ is given by the quadratic expression

$$P(u_y,v_y)=w'W_p(t_0,T)w-y'W_e(t_0,T)y. \tag{3.7}$$

The max-min solution will consequently be obtained as the solution (whenever it exists) to the following quadratic programming problem, which we denote $\bar{P}$:

$$(\bar{P})\quad \underset{y\in R^n}{\text{maximize}}\left[w'W_p(t_0,T)w-y'W_e(t_0,T)y\right]$$

subject to the constraint

$$W_p(t_0,T)w=z_0-W_e(t_0,T)y. \tag{3.8}$$

We shall need to make use of the "generalized inverse" $M^{\dagger}$ of a real matrix M which is the (unique) solution of the so called "Penrose equations."

1) $MM^{\dagger}M=M$
2) $M^{\dagger}MM^{\dagger}=M^{\dagger}$
3) $(MM^{\dagger})'=MM^{\dagger}$
4) $(M^{\dagger}M)'=M^{\dagger}M$.

Note that $M'^{\dagger}=M^{\dagger\prime}$. Also, $MM^{\dagger}$ is the projection onto $\mathcal{R}(M)$ along $\mathcal{N}(M')$ (where $\mathcal{N}$ denotes null space). In addition, we will make use of the fact that $\mathcal{N}(M')=\mathcal{N}(M^{\dagger})$ and of the fact that if M is symmetric so is $M^{\dagger}$. For a system of linear equations

$$x=My \tag{3.9}$$

the general solution is expressible as

$$y=M^{\dagger}x+\mathcal{N}(M). \tag{3.10}$$

(For a general theory of generalized inverses and computational methods, the reader is referred to [11].)

In view of (3.10), the general solution to (3.8) is given by

$$w=W_p^{\dagger}(z_0-W_ey)+\mathcal{N}(W_p) \tag{3.11}$$

when (2.4) and (2.5) hold. Here we have simplified notation temporarily and written W_p in place of $W_p(t_0,T)$ and similarly for W_e. Upon employing 1) and 2), the substitution of (3.11) into (3.7) yields the quadratic form

$$P(u_y,v_y)=z_0'W_p^{\dagger}z_0-2y'W_eW_p^{\dagger}z_0+y'\left(W_eW_p^{\dagger}W_e-W_e\right)y. \tag{3.12}$$

Upon employing standard arguments of optimization theory it is readily verified that a necessary and sufficient condition for existence of a maximum of (3.12) is that

$$W_eW_p^{\dagger}z_0\in\mathcal{R}\left(W_eW_p^{\dagger}W_e-W_e\right) \tag{3.13}$$

and

$$W_eW_p^{\dagger}W_e-W_e\leqslant 0\quad\text{(nonpositive definite)}. \tag{3.14}$$

When (2.5) holds, we also have, in view of the projection property of the generalized inverse, that

$$W_e=W_pW_p^{\dagger}W_e. \tag{3.15}$$

We will also make use of the following lemmas.

Lemma 3.1: Let (2.5) hold. Then

$$\mathcal{N}\left(W_e W_p^\dagger W_e - W_e\right) = \mathcal{N}(W_e) + \mathcal{N}(W_e - W_p). \quad (3.16)$$

Proof: Clearly, $\mathcal{N}(W_e) \subset \mathcal{N}(W_e W_p^\dagger W_e - W_e)$. Upon employing (3.15) it is also evident that $\mathcal{N}(W_e - W_p) \subset \mathcal{N}(W_e W_p^\dagger (W_e - W_p)) = \mathcal{N}(W_e W_p^\dagger W_e - W_e)$. Hence, the left side of (3.16) contains the right.

Conversely, assume $g \in \mathcal{N}(W_e W_p^\dagger W_e - W_e)$. Then upon employing (3.15) it follows that $q \in \mathcal{N}(W_e - W_p)$, where $q = W_p^\dagger W_e g$. Again, with (3.15), we then obtain $W_p q = W_e g$ and it is readily verified that $q - g \in \mathcal{N}(W_e)$, and finally, $g \in \mathcal{N}(W_e - W_p) + \mathcal{N}(W_e)$. Since g was chosen arbitrarily, this concludes the proof. □

Lemma 3.2: Let (2.4) and (2.5) hold. A necessary and sufficient condition that (3.13) holds is that

$$z_0 \in \mathcal{R}(W_e - W_p). \quad (3.17)$$

Proof: Sufficiency follows directly from (2.5) and (3.15). To prove necessity assume that (3.13) holds. Then Lemma 3.1 implies that $g' W_e W_p^\dagger z_0 = 0$ for each $g \in \mathcal{N}(W_e - W_p)$. But then $W_e g = W_p g$, and hence

$$0 = g' W_e W_p z_0 = g' W_p W_p^\dagger z_0 = g' z_0$$

where the last equality follows from (2.4). Thus, z_0 is orthogonal to $\mathcal{N}(W_e - W_p)$ which implies (3.17). □

Assume now that (2.4), (2.5), and (3.17) hold. Upon differentiating (3.12) we find that any vector y^* satisfying

$$\left(W_e W_p^\dagger W_e - W_e\right) y^* = W_e W_p^\dagger z_0 \quad (3.18)$$

maximizes (3.12). One particular solution of (3.18) is given by $y^* = (W_e - W_p)^\dagger z_0$. Indeed, we have

$$\left(W_e W_p^\dagger W_e - W_e\right)\left(W_e - W_p\right)^\dagger z_0 = W_e W_p^\dagger \left(W_e - W_p\right) \cdot \left(W_e - W_p\right)^\dagger z_0 = W_e W_p^\dagger z_0$$

where the first equality follows by (3.15), and the second equality is due to (3.17) and the fact that for any matrix M we have $MM^\dagger x = x$ for all $x \in \mathcal{R}(M)$. Thus, recalling that by Lemma 3.1, $\mathcal{N}(W_e W_p^\dagger W_e - W_e) = \mathcal{N}(W_e) + \mathcal{N}(W_e - W_p)$, we have that the general solution of (3.18), and therefore to problem ($\bar{P}$), is given by

$$y^* = \left(W_e - W_p\right)^\dagger z_0 + \mathcal{N}(W_e) + \mathcal{N}(W_e - W_p). \quad (3.19)$$

Upon subsitution of (3.19) into (3.11) we obtain

$$w^* = W_p^\dagger \left(z_0 - W_e\left(W_e - W_p\right)^\dagger z_0 + W_e \mathcal{N}(W_e - W_p)\right) + \mathcal{N}(W_p). \quad (3.20)$$

Noting that $W_e(W_e - W_p)^\dagger z_0 = (W_e - W_p)(W_e - W_p)^\dagger z_0 + W_p(W_e - W_p)^\dagger z_0 = z_0 + W_p(W_e - W_p)^\dagger z_0$ and that $W_e \mathcal{N}(W_e - W_p) = W_p \mathcal{N}(W_e - W_p)$, we obtain

$$w^* = -W_p^\dagger W_p \left(\left(W_e - W_p\right)^\dagger z_0 + \mathcal{N}(W_e - W_p)\right) + \mathcal{N}(W_p). \quad (3.21)$$

Substitution of (3.21) and (3.19) into (3.4) and (3.6), respectively, yields the following formulas for the optimal controls in problem (P):

$$u^*(t) = \tilde{B}_p'(t)\left[W_p^\dagger W_p\left(\left(W_e - W_p\right)^\dagger z_0 + \mathcal{N}(W_e - W_p)\right) + \mathcal{N}(W_p)\right] \quad (3.22)$$

$$v^*(t) = \tilde{B}_e'(t)\left[\left(W_e - W_p\right)^\dagger z_0 + \mathcal{N}(W_e) + \mathcal{N}(W_e - W_p)\right]. \quad (3.23)$$

Remark 3.3: For the one-player case (i.e., $B_e(t) \equiv 0$), formula (3.22) reduces to the optimal control law in [2] and [3] for the problem of driving the event (t_0, z_0) to $(T, 0)$ with minimum energy. Formula (3.23), however, has no analog in the one-player theory. Notice also that for existence of solutions in the one player minimum energy problems, conditions beyond controllability [such as (3.14)] do not arise since the individual Grammians are always semidefinite.

We shall now compute the optimal value $P(u^*, v^*)$ of the performance criterion. First, upon employing (3.17) and (3.15) we observe that $y^{*\prime}(W_e W_p^\dagger W_e - W_e) y^* = y^{*\prime} W_e W_p^\dagger z_0$, and hence after substituting into (3.12), we obtain

$$P(u^*, v^*) = z_0' W_p^\dagger z_0 - z_0' W_p^\dagger W_e y^*. \quad (3.24)$$

Next note that in view of (2.4) and (3.17) $z_0' W_p^\dagger W_e \cdot (\mathcal{N}(W_e) + \mathcal{N}(W_e - W_p)) = z_0' W_p^\dagger W_e \mathcal{N}(W_e - W_p) = z_0' W_p^\dagger W_p \mathcal{N}(W_e - W_p) = z_0' \mathcal{N}(W_e - W_p) = 0$, and hence we obtain $P(u^*, v^*) = z_0' W_p^\dagger z_0 - z_0' W_p^\dagger W_e (W_e - W_p)^\dagger z_0 = z_0' W_p^\dagger (z_0 - (W_e - W_p)(W_e - W_p)^\dagger z_0 - W_p(W_e - W_p)^\dagger z_0 = z_0' W_p^\dagger W_p (W_p - W_e)^\dagger z_0$, and finally

$$P(u^*, v^*) = z_0'(W_p - W_e)^\dagger z_0. \quad (3.25)$$

It is interesting to observe that $P(u^*, v^*) > 0$ for all $z_0 \neq 0$. Indeed, choosing $y = 0$ in (3.12) gives $P(u_0, v_0) = z_0' W_p^\dagger z_0$. Due to the fact that $W_p \geqslant 0$, so is $W_p^\dagger$ and hence $P(u^*, v^*) \geqslant P(u_0, v_0) \geqslant 0$. Furthermore, $P(u_0, v_0) = 0$ if and only if $z_0 \in \mathcal{N}(W_p^\dagger) = \mathcal{N}(W_p)$. But by (2.5), $\mathcal{N}(W_p) \subset \mathcal{N}(W_e)$ and hence, if $P(u_0, v_0) = 0$ then $z_0 \in \mathcal{N}(W_p - W_e)$ which together with (3.17) implies $z_0 = 0$. Thus, given that $z_0 \neq 0$, the evader, by declaring v^*, is assured that any pursuer control which steers (t_0, z_0) to $(T, 0)$ uses strictly more energy than $\int_{t_0}^T \|v^*(t)\|^2 dt$.

Our observations are summarized in the following.

Theorem 3.4: Consider a standard system (1.4) with $z_0 \neq 0$ and let $T > t_0$ be such that (2.4) and (2.5) both hold. A necessary and sufficient condition that there exist optimal controls u^* and v^* in problem (P) for the pursuer and evader, respectively, is that (3.14) and (3.17) hold. These controls are unique up to subspace translations and are given by (3.22) and (3.23). Formulas (3.22) and (3.23) uniquely determine $P(u^*, v^*)$ via (3.25). Furthermore, $P(u^*, v^*) > 0$. (In formulas (3.14), (3.17), (3.22), (3.23) and (3.25) we have abbreviated $W_e \triangleq W_e(t_0, T)$, $W_p \triangleq W_p(t_0, T)$.)

Remark 3.5: Given that (2.5) holds, one can readily verify that

$$\mathcal{R}(W_e - W_p) \subset \mathcal{R}(W_p) \tag{3.26}$$

and that

$$\mathcal{R}(W_e - W_p) = \mathcal{R}(W_p) \tag{3.27}$$

if and only if in addition $\mathcal{R}(W_e) \subset \mathcal{R}(W_e - W_p)$ holds. Formula (3.26) implies that, given (2.5), the set of events (t_0, z_0) for which there exists a solution to problem (P) is contained in the set of max-min controllable events, which of course was to be expected. However, (3.27) indicates that max-min controllability in general does not in itself guarantee existence of an optimal solution to the max-min control problem (P). Hence, there may exist events which are max-min controllable but for which an optimal solution does not exist. This interesting situation deserves some further investigation.

We now turn our attention toward deriving a more tractable characterization of the semidefiniteness condition (3.14). To this end we require the following lemmas.

Lemma 3.6: If A and B are symmetric and $A>0$, $B>0$ (i.e., positive definite matrices), then $A-B \geqslant 0$ holds if and only if $A^{-1}-B^{-1} \leqslant 0$.

Proof: There exist a real nonsingular matrix R such that $R'AR=\Lambda$ and $R'BR=I$ where I is the identity matrix and Λ is a real diagonal matrix (see, e.g., [13, p. 58]). If $A-B \geqslant 0$ then $R'AR - R'BR = \Lambda - I \geqslant 0$ implies that the diagonal elements of Λ are all $\geqslant 1$ and therefore, $I-\Lambda^{-1} \geqslant 0$. Thus, $RR' - R\Lambda^{-1}R' \geqslant 0$. Since $RR' = B^{-1}$ and $R\Lambda^{-1}R' = A^{-1}$, it follows that $A^{-1} - B^{-1} \leqslant 0$. □

Lemma 3.7: The following all hold.

1) $W_p^\dagger - W_e^\dagger \leqslant 0$ implies (3.14).

2) $W_p - W_e \geqslant 0$ implies $W_p^\dagger - W_e^\dagger \leqslant 0$ if and only if $\mathcal{R}(W_e) = \mathcal{R}(W_p)$.

3)[1] $W_p - W_e \geqslant 0$ implies (3.14) if $\mathcal{R}(W_e) = \mathcal{R}(W_p)$.

Proof: 1) Write $W_e = W_e W_e^\dagger W_e$ so that $W_e W_p^\dagger W_e - W_e = W_e(W_p^\dagger - W_e^\dagger)W_e$. Hence, $W_p^\dagger - W_e^\dagger \leqslant 0$ implies (3.14).

2) Assume first that $W_p - W_e \geqslant 0$ and that $\mathcal{V} \triangleq \mathcal{R}(W_e) = \mathcal{R}(W_p)$. In view of the symmetry of W_e and W_p it is clear that $(W_p - W_e)x = (W_p^\dagger - W_e^\dagger)x = 0$ for all $x \in \mathcal{V}^\perp$ (where $\mathcal{V}^\perp$ is the orthogonal complement of $\mathcal{V}$). Hence it suffices to show that $x'(W_p^\dagger - W_e^\dagger)x \leqslant 0$ for all $x \in \mathcal{V}$. Let $p = \dim(\mathcal{V})$ and let V be an $(n \times p)$ basis matrix for $\mathcal{V}$ (i.e., for each $x \in \mathcal{V}$ there is a unique $y \in R^p$ such that $x = Vy$). Then $W_p - W_e \geqslant 0$ is equivalent to $V'W_pV - V'W_eV \geqslant 0$. Since both $V'W_pV$ and $V'W_eV$ are nonsingular and hence positive definite, application of Lemma 3.6 completes the proof. Conversely, suppose $W_p - W_e \geqslant 0$ and $W_p^\dagger - W_e^\dagger \leqslant 0$. Let $x \in \mathcal{N}(W_e) = \mathcal{N}(W_e^\dagger)$. Then $x'W_px \geqslant 0$ and $x'W_p^\dagger x \leqslant 0$, and since both W_p and $W_p^\dagger$ are nonnegative we conclude that $x'W_px = 0$, whence $x \in \mathcal{N}(W_p)$. Consequently, $\mathcal{N}(W_e) \subset \mathcal{N}(W_p)$ and hence $\mathcal{R}(W_p) \subset \mathcal{R}(W_e)$. Since the opposite inclusion can be verified similarly, the proof is complete.

3) An immediate consequence of 1) and 2). □

Combining Theorem 3.4 and Lemma 3.7-3) yields the following result upon noting that $W_p - W_e \geqslant 0$ implies (2.5).

Corollary 3.8: Consider system (1.4) with $z_0 \neq 0$ and let $T > t_0$ be such that (2.4) holds. In addition assume that $\mathcal{R}(W_e) = \mathcal{R}(W_p)$ (where W_e and W_p are abbreviations as in Theorem 3.4). Then a sufficient condition for the existence of an optimal solution to problem (P) is that (3.17) holds and $W_p - W_e \geqslant 0$.

In Lemma 3.7 we were able to replace condition (3.14) with the simpler condition $W_p - W_e \geqslant 0$, provided we assumed $\mathcal{R}(W_e) = \mathcal{R}(W_p)$. In the next lemma we will prove that the same simplification can be accomplished if instead we assume that $\mathcal{R}(W_p) = R^n$.

Lemma 3.9: Assume $\mathcal{R}(W_p) = R^n$ (i.e., $W_p > 0$). Then (3.14) holds if and only if $W_p - W_e \geqslant 0$.

Proof: There exists a nonsingular real matrix R such that $R'W_eR = \Lambda$ and $R'W_pR = I$ where Λ is a diagonal matrix with nonnegative entries (recall the proof of Lemma 3.6). Now $W_p - W_e = R^{-1\prime}(I - \Lambda)R^{-1}$. Also, $W_eW_p^{-1}W_e - W_e = R^{-1\prime}(\Lambda^2 - \Lambda)R^{-1}$. The proof is completed upon noting that $\Lambda^2 - \Lambda \leqslant 0$ if and only if $I - \Lambda \geqslant 0$. □

In view of the previous lemma we have the following additional corollary to Theorem 3.4.

Corollary 3.10: Consider system (1.4) with $\mathcal{R}(W_p) = R^n$ (where W_p and W_e are abbreviations as in Theorem 3.4). Then the following hold.

1) A necessary and sufficient condition that there exists an optimal solution to problem (P) for every event in $\{t_0 \times R^n\}$ is that $W_p - W_e > 0$, in which case the optimal pursuer control [as given by (3.22)] is unique.

2) If $W_p - W_e$ is singular, then a necessary and sufficient condition that there exists an optimal solution to problem (P) for an event (t_0, z_0) is that $z_0 \in \mathcal{R}(W_e - W_p)$ and $W_p - W_e \geqslant 0$.

Remark 3.11: In [4] problem (P) was solved via a variational penalty function method for the special case where $W_p - W_e > 0$. The authors of [4] suggested that $W_p - W_e > 0$ meant that the pursuer was "more controllable" than the evader, conveying the intuitive idea that a kind of (max-min) controllability property exists. Indeed, $W_p - W_e \geqslant 0$ implies (2.5). From the development in the present paper, however, we see that max-min controllability can hold in the *absence* of definiteness conditions on $W_p - W_e$. Consider, for example, $W_p = I$ and $W_e = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

Remark 3.12: Due to the Cauchy–Binet theorem (see, e.g., [16, p. 15]), $W_p - W_e \geqslant 0$ implies $\det(W_p) \geqslant \det(W_e)$.

IV. Further Remarks

In this section we shall address ourselves to the comparison of our results in the previous sections with certain results and concepts in the existing body of differential game literature.

[1]There is no "only if" in 3); see Lemma 3.9.

We have refrained from calling (P) a differential game due to the information structure which we have imposed on the problem. While this "open-loop" structure finds application in many engineering and economic models of competitive situations, open-loop strategies are specializations of more general types of strategies which can be found in the literature. Two specific approaches made reference to below are Hájek [8], [9], and Isaacs [7].

Hájek in [8] considered strategies to be "snap decision rules"; i.e., a player's control depends instantaneously upon the opposing player's control. Hájek considered a question which might be termed "strategic max-min controllability," or the pursuer's ability to steer z_0 to 0 in a variant of system (1.4) in the presence of any evader strategy. For the case of (possibly time varying) control and state constraints he showed that strategic max-min controllability of an event is equivalent to conventional controllability in a certain associated one-player linear system. Controllability in this system depends on the Pontryagin difference of the players' control constraints, and after some interpretation it can be shown, as one would expect, that this system's controllability implies max-min controllability in our sense. Extensions to other types of strategies and targets as well as strategy design may be found in [8] and [9]. A study of the relationship between the results of Hájek and those given in the present paper may prove worthwhile, in view of the overall desirability of bringing control theoretic tools to bear on differential games.

If we allow the evader some measurements of the state, the conditions derived in Section II may no longer be sufficient for the pursuer to force the initial state to the origin in system (1.4) even in the autonomous case. To see that this is the case consider the following simple example.

Example 4.1: Consider system (1.1) with

$$A=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad B_e=\begin{pmatrix}0\\-1\end{pmatrix},\quad B_p=\begin{pmatrix}1\\1\end{pmatrix}.$$

The initial event is $\left(0,\begin{pmatrix}1\\2\end{pmatrix}\right)$ and it is easily seen that conditions (2.12) and (2.13) hold. Hence, the initial event is max-min controllable in system (1.1). If, however, the evader employs the linear *feedback* rule $v(t)=Kx(t)$ with $K=(0,1)$, then no pursuer control can steer the initial state to the origin in finite time.

Example 4.1 indicates that under certain conditions, the evader has the capability, by employing a constant linear feedback control, to destory the max-min controllability property and hence prevent the pursuer from ever being able to force capture. We will now show that this capability is, indeed, quite general.

Consider system (1.1) with A, B_p and B_e constant real matrices and assume

$$\mathfrak{R}([A|B_e])\subset\mathfrak{R}([A|B_p]). \tag{4.1}$$

We now state the following problem: Under what conditions does there exist an $m_e\times n$ constant real matrix F such that

$$\mathfrak{R}([A+B_eF|B_e])\not\subset\mathfrak{R}([A+B_eF|B_p]). \tag{4.2}$$

Clearly, if the evader can find F such that (4.2) holds then the pursuer cannot force capture for any initial state $x_0\neq 0$. We shall require the following lemma (see [14, p. 45] for details).

Lemma 4.2: Let X and Y be finite-dimensional linear spaces, and let $G:Y\to X$ be a linear map. Then for a linear map $D:X\to X$ there exists a map $C:X\to Y$ such that $D=GC$ if and only if $\mathfrak{R}(D)\subset\mathfrak{R}(G)$.

In view of Lemma 4.2 we can rephrase the aforementioned question as follows: given that (4.1) holds, characterize the existence of a constant $(n\times n)$ matrix A_e such that

$$\mathfrak{R}(A_e)\subset\mathfrak{R}(B_e) \tag{4.3}$$

and

$$\mathfrak{R}([A+A_e|B_e])\not\subset\mathfrak{R}([A+A_e|B_p]). \tag{4.4}$$

Theorem 4.3: Given system (1.1) with A, B_p, B_e constant real matrices such that (4.1) holds and $B_p\neq 0$. A necessary and sufficient condition for the existence of a matrix A_e such that (4.3) and (4.4) hold is

$$\mathfrak{R}(B_e)\not\subset\mathfrak{R}(B_p). \tag{4.5}$$

Proof: Clearly $\mathfrak{R}([A+A_e|B_e])=\mathfrak{R}([A|B_e])$ for every A_e which satisfies (4.3). If $\mathfrak{R}(B_e)\subset\mathfrak{R}(B_p)$, then $\mathfrak{R}(A_e)\subset\mathfrak{R}(B_p)$ and consequently $\mathfrak{R}([A+A_e|B_p])=\mathfrak{R}([A|B_p])$ and hence A_e cannot be found to satisfy (4.4). Conversely, assume there exists a subspace $\mathcal{V}\subset\mathfrak{R}(B_e)$ such that $\mathcal{V}\neq 0$, $\mathcal{V}\cap\mathfrak{R}(B_p)=0$. Write $\mathfrak{R}^n=\mathcal{V}\oplus\mathcal{W}$ for a subspace $\mathcal{W}$ which satisfies $\mathfrak{R}(B_p)\subset\mathcal{W}$. Let P be the projection of R^n on $\mathcal{V}$ along $\mathcal{W}$, and define $A_e=-PA$. This A_e clearly satisfies (4.3) and $A+A_e=(I-P)A$ is such that $\mathfrak{R}(A+A_e)\subset\mathcal{W}$. Hence, since $\mathfrak{R}(B_p)\subset\mathcal{W}$, it follows that $\mathfrak{R}([A+A_e|B_p])\subset\mathcal{W}$. Since $\mathcal{V}\not\subset\mathcal{W}$, (4.4) follows. □

Theorem 4.3 shows that when the evader has access to state measurements, he can prevent capture for any initial state as long as (4.5) holds. Hence, assuming that feedback is allowed, the only cases of potential interest are those in which $\mathfrak{R}(B_e)\subset\mathfrak{R}(B_p)$. But then the pursuer can essentially undo instantly all the evader's actions and the problem loses many of its more interesting features.

For the nonantonomous case [i.e., system (1.4)], allowing the evader only *one observation of the state* may prevent the pursuer from steering (t_0,z_0) to the origin even though (2.4) and (2.5) (i.e., max-min controllability) hold. This phenomenon is illustrated in the following example.

Example 4.4: Let the system matrices be given by

$$B_e(t)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\qquad \text{for all } t\geq 0,$$

while

$$B_p(t)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\qquad t\in[0,1]$$

and

$$B_p(t)=B_e(t),\qquad \text{for all } t\in(1,2]$$

and

$$B_p(t)=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\qquad \text{for all } t>2.$$

One can readily check that $\mathfrak{R}[W_p\,(0,3)]=R^3$, and hence all events $(0,z_0)$ are max-min controllable in time 3. Note, however, that $\mathfrak{R}[W_p(\tau,3)]\not\supset\mathfrak{R}[W_e(\tau,3)]$ for $\tau\geqslant 2$. Hence, there are no max-min controllable events of the form $(2,z)$. Thus, if we allow the evader to make one measurement of the state at time 2 the pursuer cannot force the solution of (1.4) to the origin.

Notice that the property exhibited in Example 4.4 is strictly a "nonautonomous" phenomenon. To see this, one can readily verify that a sufficient condition [beyond (2.4) and (2.5)] for preclusion is $\mathfrak{R}[W_e(\tau,T)]\subset\mathfrak{R}[W_p(\tau,T)]$ for all $\tau\in(t_0,T)$, and that the autonomous version of (1.1) satisfies this condition. In fact, the same comment holds if we allow the evader any finite number of observations of the state.

One further digression on the subject of feedback is in order. The optimal controls u^* and v^*, given by (3.22) and (3.23), respectively, may be written in linear feedback form. Following [3, p. 48], we obtain

$$\begin{aligned}u^*(t)=u^*(t,z^*(t))&=\tilde{B}_p'(t)\Big(\big(W_p^\dagger(\tau,T)W_p(\tau,T)\\&\quad\cdot(W_e(\tau,T)-W_p(\tau,T))^\dagger z^*(t)\\&\quad+\mathfrak{N}[W_e(\tau,T)-W_p(\tau,T)]\big)+\mathfrak{N}[W_p(\tau,T)]\Big)\qquad(4.6)\end{aligned}$$

and

$$\begin{aligned}v^*(t)=v^*(t,z^*(t))&=\tilde{B}_e'(t)\Big((W_e(\tau,T)-W_p(\tau,T))^\dagger\\&\quad\cdot z^*(t)+\mathfrak{N}[W_e(\tau,T)]\\&\quad+\mathfrak{N}[W_e(\tau,T)-W_p(\tau,T)]\Big)\qquad(4.7)\end{aligned}$$

where $z^*(t)$ is the solution to (1.4) associated with the controls given in (3.22) and (3.23). While (4.6) and (4.7) provide a nonunique closed-loop synthesis of u^* and v^*, these formulas do *not* provide optimal feedback control laws; indeed, Examples 4.1 and 4.4 show this.

We now shall turn to certain questions of values and saddle points. First, consider problem (P) and suppose the $z_0\in\mathfrak{R}[W_p(t_0,T)]$ and $\mathfrak{R}[W_e(t_0,T)]=\mathfrak{R}[W_p(t_0,T)]$. We can define a "min-max" game: The minimizer (i.e., the chooser of u) declares his control first. The maximizer then chooses a control v such that z_0 is steered to 0. In view of conditions (3.14) and (3.17) we see that the max-min and min-max simultaneously exist if and only if $z_0=0$ and $W_e(t_0,T)=W_p(t_0,T)$. Thus, value in an open-loop sense for the restricted endpoint problem discussed herein exists only for a very special situation.

We shall conclude by noting the connection between our results in Section III, the results of Ho, Bryson, and Baron [4], and Isaacs' approach to differential games. Consider a differential game with dynamics (1.4), payoff

$$P(u,v)=\alpha\|z(T)\|^2+\int_{t_0}^T\|v(t)\|^2dt-\int_{t_0}^T\|u(t)\|^2dt\qquad(4.8)$$

where $T>t_0$ is given, $\alpha>0$, and where there are no terminal restraints. Following [4], we see that a sufficient condition for the solvability of the Isaacs' equation (and therefore for the existence of value and a closed-loop saddle point) for every $\alpha>0$ is that $W_p(t_0,T)-W_e(t_0,T)>0$. Upon comparing the formulas in [4] for the optimal controls in the above differential game with formulas (3.22) and (3.23), we see that the latter controls are true limiting cases as $\alpha\to\infty$ of the former. Hence, by adding the "penalty" term $\alpha\|z(T)\|^2$ to (1.2) and letting $\alpha\to\infty$ in the free endpoint game, we have that the value $V(\alpha)$ converges to $P(u^*,v^*)$, our max-min. The approximating games have closed-loop saddle points, while (P), in general, has not even a value in Isaacs' sense. Arguing similarly, one can also prove that if v is required to lie in a ball $\{w\in R^e:\|w\|\leqslant\beta\}$ then there exists a feedback law $u(z)$ for the pursuer such that against any admissible evader feedback law the associated solution of (1.4) satisfies $\|z(T)\|\leqslant\gamma$, where $\gamma>0$ depends on β. As is seen in Example 4.1, however, for the case $\gamma=0$ there might not be such a pursuer feedback law for *any* $\beta>0$.

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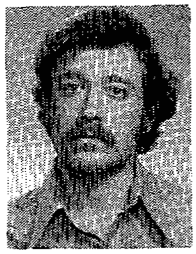

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